

Stability of digital discrete time controllers for water systems

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Abstract All water systems with automatic controllers implemented by programs running on digital computers are sampled data systems. The water systems themselves are often non linear systems. This complicates the analysis of the stability of the combined system. In this paper we apply the second method of Lyapunov to a trunk sewer with a gate regulated by a discrete controller. This is intended as a feasibility study of a theoretical approach to the design and stability analysis of such controllers.

Keywords: Lyapunov function; discrete controller; sewer system.

1 INTRODUCTION

Many papers have been published on the application of Real Time Control (RTC) in the fields of urban drainage, treatment plant operation, irrigation engineering and water supply systems [Schilling, 1994; Puig et al., 2009; Campisano et al., 2005; Marinaki and Papageorgiou, 2005; Schütze et al., 2004; Pleau et al., 2005]. In urban drainage systems, RTC shows considerable promise as a cost-attractive technique to reduce flooding in urban areas, overflow discharges from the wastewater system to the receiving water bodies or to mitigate flow peaks to the treatment plant. In-sewer moveable gates can activate available in-line storage capacity in sewers and storage tanks [Charron et al., 2001; Pleau et al., 2001] and so contribute to these objectives. Even one sewer with one gate can be of theoretical interest, something that may not be immediately obvious to specialists in control theory. This is a pity because solutions to questions of theoretical interest will become of more and more importance to practitioners. We hope that this article will function as a bridge between the two worlds. We use a simple example, based on the system described in Campisano and Modica [2002]. The system consists of a trunk sewer that can be used for storage. This use is controlled by a movable gate. The stage-discharge relation of the gate is non-linear and the controller is discrete with a time step of 60 seconds. We will give a description of the system in the next section. We then develop a controller and finally we draw some conclusions. The application of Lyapunov functions to canal control is well developed for the continuous case, see for example Coron et al. [2007] or Bastin et al. [2008] and references therein. Other treatments of canal control design can be found in Campisano and Modica [2002].

2 MODEL FORMULATION AND PARAMETER DEFINITION.

We consider a trunk sewer of 1.2 kilometers in length with a constant slope of 0.1% and a rectangular cross section, 1.0 wide by 2.0 meters high. The sluice gate is 1.0 km

downstream of the upstream entrance. The time step between two consecutive control actions is 60 seconds. We wish to maintain the water depth immediately upstream of the gate at 1.6 meters. This should fill approximately 80% of the total volume of the sewer. We assume that the sill of the gate is level with the sewer floor and that the gate leaf is lowered from a slot in the sewer ceiling.

We have the following variables: the gate opening a (distance from sill to bottom of gate leaf) and the depth y at the measurement location. We treat the flow Q_e at the entrance to the trunk sewer and the target depth $y_t = 1.6$ m at the measurement location as fixed parameters. Other parameters of the system are: sewer height $y_{\max} = 2$ m, sewer width $w = 1$ m, sewer slope $S_0 = 0.001$, distance from entry point to gate $x = 1000$ m, the discharge $Q_u = 1.860$ m³/s for which y_t is the uniform flow depth, gate width $w_g = 1$ m, minimum gate opening $\hat{a}_{\min} = 0.1$ m, maximum gate opening $a_{\max} = 2.0$ m, gate contraction coefficient C_c , maximum gate movement rate $v_a = 0.005$ m/s, control time step $\Delta t = 60$ s.

3 ASSUMPTIONS

For a sensible discussion of a controller for the gate in the trunk sewer we need to make three sets of assumptions. The first set consists of fundamental assumptions, without them the problem is unsolvable. The second set contains necessary technical assumptions, they are basic to the arguments used in this paper. The third set are simplifying assumptions that we feel might be dropped, but are used to allow us to keep this paper to a manageable length. We will give each set in turn. To give the arguments in mathematical form we introduce the following terminology and notation. When we use the term state of the system we will mean the position of the gate together with the full physical state of the canal. This includes, but is not limited to, depths over the full cross section at all points along the sewer and velocities over the full cross section at all points of the canal.

Let S_1 be the set of all initial states for the canal that make physical sense, while this is not mathematically precise it avoids an overly theoretical approach that allows for instance every possible depth and velocity field. We would like to consider the set of all states that the canal can reach, but that set depends on what inflow patterns can occur and how the gate can move. It may also depend on the initial state. We will use A_Q to represent the set of realizable inflows given as functions of time and A_a as the set of realizable gate openings as a function of time. Now we can define, for each $s_0 \in S_1$, the set $S(s_0, A_Q, A_a)$ of all states that the canal can reach. Assume that $S(s_0, A_Q, A_a) \subset S_1$. For any initial state s_0 and any functions $Q_0 \in A_Q$ and $a_0 \in A_a$ and any starting time t_0 we can now define the function $s(t, s_0, Q_0, a_0) : [t_0, \infty[\rightarrow S(s_0, A_Q, A_a)$ that represents the evolution of the system state in time. On $S(s_0, A_Q, A_a)$ we define several functions. For $s \in S(s_0, A_Q, A_a)$ we define projections P to extract relevant information from the state, the value of $P_a(s)$ is the gate opening for that state, the value of $P_{us}(s)$ is the water depth at the measurement location upstream of the gate, the value of $P_{ds}(s)$ is the water depth at the measurement location downstream of the gate and $P_V(s)$ is the volume of sewage in the sewer upstream of the gate. We will call a state a stationary state when the discharge through a cross section is constant and independent of time along the canal. The function s is causal, in other words $s(t)$ only depends on past values of the functions Q_0 and a_0 up to time and not on future values.

3.1 Fundamental assumptions

We will provide these as a numbered list.

1. The gate opening can be kept fixed: if $s_0 \in S_1$ then the function $a(t) = P_a(s_0)$ is always in A_a .
2. For a fixed gate opening and a constant inflow into the system there is a stationary state and in time the system will evolve towards that stationary state: if $Q_0(t) = Q_e$ with Q_e constant is in A_Q and we take $a_0(t) = P_a(s_0)$ then for all $s_0 \in S_1$ there is a $s_{\text{stat}}(s_0, Q_0, P_a(s_0))$ such that

$$\lim_{t \rightarrow \infty} P_{\text{us}}(s(t, s_0, Q_0, a_0)) = P_{\text{us}}(s_{\text{stat}}(s_0, Q_0, P_a(s_0)))$$

$$\lim_{t \rightarrow \infty} P_{\text{ds}}(s(t, s_0, Q_0, a_0)) = P_{\text{ds}}(s_{\text{stat}}(s_0, Q_0, P_a(s_0)))$$

$$\lim_{t \rightarrow \infty} P_V(s(t, s_0, Q_0, a_0)) = P_V(s_{\text{stat}}(s_0, Q_0, P_a(s_0)))$$

3. For a given gate opening and discharge the corresponding stationary state is unique, so if $P_a(s_0) = P_a(s'_0)$ then $s_{\text{stat}}(s_0, Q_0, P_a(s_0)) = s_{\text{stat}}(s'_0, Q_0, P_a(s_0))$. In other words, for the stationary state volume, upstream depth and downstream depth depend only on discharge and gate opening, so for the stationary state corresponding to Q_0, a_0 we can define the stored volume $V_{\text{stat}}(Q_0, a_0)$, the upstream depth $y_{\text{stat}}^{(\text{us})}(Q_0, a_0)$ and the downstream depth $y_{\text{stat}}^{(\text{ds})}(Q_0, a_0)$.
4. For each entrance flow there is a uniform flow depth and at that depth there is free surface flow.
5. If in a stationary state s the gate does not touch the water then $P_{\text{us}}(s) = P_{\text{ds}}(s)$ and these depths depend only on Q_0 .

3.2 Necessary technical assumptions

The following assumptions are needed to guarantee reasonable system behavior.

6. There are upper and lower limits on the entrance flow: for all $Q(t) \in A_Q$ and for all t we have $Q_{e,\text{min}} \leq Q(t) \leq Q_{e,\text{max}}$.
7. There is a depth y_{free} such that below that depth the flow in the sewer is free surface flow.
8. There are upper and lower limits on the gate opening and the lower limit is strictly greater than zero: for all $a(t) \in A_a$ and for all t we have $a_{\text{min}} \leq a(t) \leq a_{\text{max}}$.
9. Between the gate and the downstream end of the trunk sewer there is at least one point where the depth equals the uniform flow depth.
10. The gate only influences the flow when it actually touches the water and there is no jump in the discharge or the levels when the gate enters or leaves the water: $P_a(s) \geq P_{\text{us}}(s)$ implies $P_{\text{us}}(s) = P_{\text{ds}}(s)$.
11. There is a friction formula for stationary flow, so there is a function S_f such that for the uniform flow depth y_0 we have $S_f(y_0, Q_0) = S_0$.
12. If we take one argument fixed then $S_f(y_0, Q_0)$ is an invertible function of the other argument, so we can define uniform flow discharge $Q_u(y_0, S_0)$ for given y_0, S_0 and uniform flow depth $y_u(Q_0, S_0)$ for given y_0, S_0 such that $S_f(y_0, Q_u(y_0, S_0)) = S_0$ and $S_f(y_u(Q_0, S_0), Q_0) = S_0$.

13. We can model the gate in the stationary case. In other words there is a function $Q_g(y_1, a, y_3)$ such that, for a stationary state s with discharge Q_0 and gate opening a_0 we have

$$Q_g(P_{us}(s), a_0, P_{ds}(s)) = Q_0$$

Note that this implies that for $a > \max(y_{us}, y_{ds})$ the function is the inverse of the formula for uniform flow depth calculation.

14. For $y > 0$ and $Q > 0$ the function $S_f(y, Q)$ satisfies

$$\frac{\partial S_f}{\partial y} < 0$$

$$\frac{\partial S_f}{\partial Q} > 0$$

$$\lim_{y \downarrow 0} S_f(y, Q) = \infty$$

$$\lim_{y \uparrow \infty} S_f(y, Q) = 0$$

$$\lim_{Q \downarrow 0} S_f(y, Q) = 0$$

Details of possible friction formulas can be found in Chow [1959], Henderson [1966] and Chaudhry [2008].

15. On the domain $[y_u(Q_{e,\min}), y_{free}] \times [a_{\min}, a_{\max}, y_u(Q_{e,\min}), y_{free}]$ the function $Q_g(y_1, a, y_3)$ is continuous and satisfies

$$\frac{\partial Q_g}{\partial y_1}(y_1, a, y_3) > 0 : a < y_1, y_3 < y_1$$

$$\frac{\partial Q_g}{\partial y_2}(y_1, a, y_3) > 0 : a < y_1, y_3 < y_1$$

$$\frac{\partial Q_g}{\partial y_3}(y_1, a, y_3) \leq 0 : a < y_1, y_3 < y_1$$

$$\lim_{z \uparrow y_1} Q_g(y_1, a, z) = 0 : a < y_1$$

and for all $\epsilon > 0$ and all $z < y_1$ there is a $\delta > 0$ such that

$$Q_g(y_1, y_1 - \delta, z) > Q_u(y_1) - \epsilon$$

Note that this implies that either there is no combination of water depths and gate opening in this range for which a hydraulic jump can exist downstream of the gate or the transition from modular to submerged flow is gradual. More information on gate discharge formulas can be found in Chow [1959], Henderson [1966], Bos [1978] and Chaudhry [2008].

Next we derive a system property from these assumptions.

Lemma 1. *If the system is stationary with non-zero flow and we have uniform flow downstream of the gate, then for $S_0 > 0$ and given all properties listed up to this point, the equation*

$$S_f(y, Q_g(y_t, a, y)) - S_0 = 0$$

has a unique solution $y \in [y_u(Q_{e,\min}), y_{free}]$ as long as $a < y_t$ and $a \in [a_{\min}, a_{\max}]$.

Proof. Take

$$f(y) = S_f(y, Q_g(y_t, a, y)) - S_0$$

for fixed y_t and a . The function $Q_g(y_t, a, y)$ does not decrease (and may increase) as y decreases. The function $S_f(y, Q)$ increases as y decreases it also increases as Q increases so f is an increasing function. As y goes to zero S_f increases without bound, so there is a $\delta_0 > 0$ such that $f(\delta_0) > 0$. As y goes to y_t we see that $Q_g(y_t, a, y)$ goes to zero and therefore $S_f(y, Q_g(y_t, a, y))$ goes to zero, so there is a $\delta_1 > 0$ such that $f(y_t - \delta_1) = 0$. The function is continuous, so there is a point where $f(y) = 0$. \square

3.3 Simplifying assumptions

The assumptions listed here are needed to simplify the formal approach, but can probably be relaxed.

16. The inflow is constant (but unknown).
17. For the given range of discharges and gate openings free surface flow implies the gate is drowned, in other words the uniform flow depth is larger than the conjugate or sequent depth for all gate openings and all upstream water depths that do not touch the top of the sewer.
18. The stationary gate formula also applies to the non-stationary case.
19. For each measurement time t_k and each water depth measurement we have three numbers, the water depth y_k at time t_k , the minimum water depth $y_{k,\min}$ during the interval $[t_{k-1}, t_k]$ and the maximum water depth $y_{k,\max}$ during the interval $[t_{k-1}, t_k]$. We have the same for the gate opening.

Remarks.

1. We need to keep in mind that if the upstream head exceeds the sewer height then we have problem because our simplifying assumption 17 no longer applies.
2. In reality water level measurements need to be filtered before sampling to remove "noise" and to remove frequencies above the Nyquist frequency for that sampling frequency, see also Young [2011].

4 OPERATION OF THE GATE

We model the discrete gate operation as follows. At time t_k the gate is given a command to change its position by u_k . If this move is possible for the given maximum movement speed and the allowed range of gate settings then the gate will be at the new position at the end of the time step. We use the notation $s(t, s_0, Q_0, \{u_k\}_{k=1}^{\infty})$ for the resulting system state as a function of time. If the sequence of commands is zero after some finite index K then we write $s(t, s_0, Q_0, \{u_k\}_{k=1}^K)$ for the resulting system state as a function of time.

4.1 A controller based on known volume

We derive an algorithm to control the gate based on knowledge of the change in stored volume per time step. We show that this algorithm will bring the system to a state with the desired water depth y_t upstream of the gate.

We show that, if we know the change in volume in a time step, for example because there is a model that is kept synchronized to the sewer using data assimilation and if we have enough information to estimate the outflow, for example from a gate flow formula, then we can estimate the actual inflow. With this estimate we can calculate whether or not the current gate opening can accommodate that discharge at the desired upstream water level and uniform flow depth downstream.

Theorem 1. *Let the system satisfy all assumptions made up to this point. In that case, for any initial state $s_0 \in S$ and any constant inflow Q_e the following controller will bring the system to the desired volume. Assume that the conditions from Lemma 1 hold and denote the solution for y of*

$$S_f(y, Q_g(y_t, a, y)) - S_0 = 0$$

by $y_{gu}(y_t, a)$, in other words the gate discharge with upstream depth y_t and downstream depth $y_{gu}(y_t, a)$ is also the discharge for which $y_{gu}(y_t, a)$ is the uniform flow depth at slope S_0 . We also define

$$Q_{gu}(y_t, a) = Q_g(y_t, a, y_{gu}(y_t, a))$$

and

$$s(t) = s\left(t, s_0, Q_e, \{u_k\}_{k=1}^K\right) : t < t_{K+1}$$

where u_k is the controller command given at time t_k and finally we define

$$s_k = s(t_k), V_k = P_V(s_k), y_k^{(us)} = P_{us}(s_k), y_k^{(ds)} = P_{ds}(s_k)$$

Now let

$$Q_{left}^{(k+1)} = V_{k+1} - V_k + Q_g\left(y_{k+1, \min}^{(us)}, a_{k+1, \min}, y_{k+1, \max}^{(us)}\right)$$

$$Q_{right}^{(k+1)} = V_{k+1} - V_k + Q_g\left(y_{k+1, \max}^{(us)}, a_{k+1, \max}, y_{k+1, \min}^{(us)}\right)$$

$$a_{left}^{(k+1)} = \max \left\{ a \in [a_{\min}, a_{\max}] : Q_{gu}(y_t, a) \leq \max \left[Q_{gu}(y_t, a_{\min}), Q_{left}^{(k+1)} \right] \right\}$$

$$a_{right}^{(k+1)} = \min \left\{ a \in [a_{\min}, a_{\max}] : Q_{gu}(y_t, a) \geq \min \left[Q_{gu}(y_t, a_{\max}), Q_{right}^{(k+1)} \right] \right\}$$

and apply the following rule: if $a_{k+1} \in [a_{left}^{(k+1)}, a_{right}^{(k+1)}]$ then $u_{k+1} = 0$ else if $a_{k+1} > a_{right}^{(k+1)}$ then $u_{k+1} = \max(-\Delta a, a_{right}^{(k+1)} - a_{k+1})$ else $u_{k+1} = \min(\Delta a, a_{left}^{(k+1)} - a_{k+1})$. This controller brings the system to a state with the correct gate opening for the actual inflow.

Proof. We define the following semi-Lyapunov function

$$L(a) = |a - a_e|$$

If $a_{k+1} \in [a_{left}, a_{right}]$ then $L(a_{k+1}) = L(a_k)$ else $L(a_{k+1}) < L(a_k)$. We still need to show that for constant gate opening the interval $[a_{left}^{(k+1)}, a_{right}^{(k+1)}]$ shrinks to one specific value. Suppose there exists a gate opening a_0 for which this does not happen. In that case there is a δ such that for all M there is an $m > M$ such that $y_{k+m+1, \max}^{(us)} - y_{k+m+1, \min}^{(us)} > \delta$ or $y_{k+m+1, \max}^{(ds)} - y_{k+m+1, \min}^{(ds)} > \delta$. We give the argument for the upstream depth

$$\begin{aligned} & y_{k+m+1, \max}^{(us)} - y_{k+m+1, \min}^{(us)} = \\ & \left| y_{k+m+1, \max}^{(us)} - y_{\text{stat}}(Q_e, a_0) + y_{\text{stat}}(Q_e, a_0) - y_{k+m+1, \min}^{(us)} \right| < \\ & \left| y_{k+m+1, \max}^{(us)} - y_{\text{stat}}(Q_e, a_0) \right| + \left| y_{k+m+1, \min}^{(us)} - y_{\text{stat}}(Q_e, a_0) \right| \end{aligned}$$

but according to our assumptions $y_k = P_y(s(t, s_k, Q_e, a(t) = a_0))$ converges to $y_{\text{final}}(Q_e, a_0)$ for all m which implies that there is an N such that

$$\left| y_{k+n+1, \text{max}}^{(\text{us})} - y_{\text{final}}(Q_e, P_a(s_k)) \right| < \delta/3$$

for all $n > N$. But then on the one hand there is an $m' > N$ such that

$$y_{k+m'+1, \text{max}}^{(\text{us})} - y_{k+m'+1, \text{min}}^{(\text{us})} > \delta$$

and on the other hand

$$y_{k+m'+1, \text{max}}^{(\text{us})} - y_{k+m'+1, \text{min}}^{(\text{us})} < 2\delta/3$$

and we have a contradiction. We can do the same for the downstream depth. \square

If the inflow varies slowly enough then the above controller should still work for a non constant entrance flow. If there are variations in inflow that average out over the controller time step then the above controller should still work. Note that the use of bounds on the outflow estimate create some tolerance for disturbances in general.

5 ESTIMATING THE VOLUME

Suppose we have a model of the canal that can approximate the volume in the canal to a given precision. If we assume this model can somehow be synchronized with the real system then the technique from Theorem 1 will work, but the width of the interval $[a_{\text{left}}, a_{\text{right}}]$ will be determined by the accuracy of our approximation of the volume. The question now becomes how to synchronize the two. In theory we could for example use a one dimensional numerical hydrodynamic model based on the St-Venant equations. When starting the controller we would then need a lead time T during which we would not move the gate. After time T we would assume stationary flow, derive Q_e from the flow at the gate and start applying the controller to both the real system and the model. We would need a form of data assimilation using the measured depths to keep the model close to the real system. To avoid incorrect controller actions we need to add an error estimate to the change in volume term.

6 DISCUSSION

We have tried to show that under suitable assumptions it is possible to derive an algorithm for a discrete controller for a gate in a trunk sewer that uses only the bulk physical properties of a system. The proof that the algorithm works is purely theoretical, but we plan to explore this approach further with computer experiments.

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