

# A method for analysing growth curves by using the logistic model with changing carrying capacity

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## Abstract

The problem of *managing resources of a limited planet* invokes the logistic model with constant carrying capacity. However, the growth of industrial systems and populations of countries with limited resources demonstrates that limits to growth are changing with new technologies. This paper presents a new method for analysing growth curves representing the logistic growth with changing carrying capacity.

**Keywords:** limits to growth, logistic growth, carrying capacity, carbon dioxide emissions

## 1 INTRODUCTION

The problem of *managing resources of a limited planet* invokes the logistic model with constant carrying capacity. This model had been used originally to analyze the growth of population, and the carrying capacity denoted the population size at which the population growth rate equals zero [Hui, 2006]. Now this model is applied in various fields of science [Meyer, 1994], and therefore the carrying capacity of a system denotes merely the value of a state variable at which its growth rate equals zero.

The assumption that carrying capacity of a system remains constant is essential from mathematical point of view: in this case we could get the analytical solution of the model equation. Moreover, we could predict the limits to growth proceeding from the observed part of the growth curve.

However, the growth of industrial systems and populations of countries with limited resources demonstrates that limits to growth are changing with new technologies [Ausubel, 1996]. The facts of such sort gave the birth to bi-logistic models of growth [Meyer, 1994]. The bi-logistic models suppose that a growth curve is a “superposition” of logistic curves with different carrying capacities.

This paper presents a new method for analyzing growth curves representing logistic growth with changing carrying capacity. It provides an analytical solution of the logistic equation when changes in carrying capacity could be represented by a continuous function (not a step-wise function like it was done earlier), and illustrates the use of the proposed method for analyzing the growth of carbon dioxide emissions.

## 2 METHODS

The logistic model of growth is given by the equation

$$N(t) = \frac{K}{1 + e^{-\alpha t - \beta}} \quad (1)$$

where  $N(t)$  is the variable characterizing the process of growth,  $t$  is time,  $K$  is its asymptotic value, characterizing the limits to growth, so called carrying capacity,  $\alpha$  characterizes the slope of the growth curve,  $\beta$  – characterizes the moment of time at which the curve reaches its midpoint (i.e.,  $0.5K$ ).

The model parameters are usually calculated to fit the data – that is, parameters are chosen to minimize the sum

$$\sum_i \frac{(N_i - D_i)^2}{W_i} \quad (2)$$

where  $N_i = N(t_i)$ ,  $D_i$  and  $W_i$  are the observed values of the system variable at the given moments of time and weight coefficients, respectively.

In many cases the asymptotic limit of growth  $K$  can be given by experts, and only  $\alpha$  and  $\beta$  are selected to minimize the difference between the observed and modeled values.

If  $N(t) \ll K$ , then the logistic model begins to resemble an exponential growth.

$$N(t) = e^{\alpha t + \beta} \quad (3)$$

For analyzing historical time-series data [Meyer, 1994] proposed the following form of the logistic model:

$$N(t) = \frac{K}{1 + e^{-\frac{\ln(81)}{\Delta t}(t - t_m)}} \quad (4)$$

where  $\Delta t$  is the length of the time interval required for  $N(t)$  to grow from 10 to 90% of  $K$ , and  $t_m$  is the midpoint of the growth process.

Meyer [1994] also proposed a so-called bi-logistic model for modeling the growth in the case when the carrying capacity is changing from  $K_1$  to  $K_2$ :

$$N(t) = \frac{K_1}{1 + e^{-\frac{\ln(81)}{\Delta t_1}(t - t_{m1})}} + \frac{K_2}{1 + e^{-\frac{\ln(81)}{\Delta t_2}(t - t_{m2})}} \quad (5)$$

Each term of the equation above forms a logistic model describing one period or “pulse” of growth as system changes from fast exponential growth to slow saturated growth. In order to identify model parameters, the data can be split into the time series representing the first pulse of growth and the time series

representing the second pulse of growth. But this method is not always good, because it is not always clear where to split the data set.

Shepherd and Stojkov [2007] proposed another method for dealing with changing carrying capacity. If carrying capacity is changing slowly, it could be considered as a positive function  $K(\varepsilon t)$  where  $t$  is time and  $\varepsilon$  an infinitesimal positive parameter. Then the differential equation representing the logistic growth model takes the form.

$$\begin{cases} \frac{dN(t, \varepsilon)}{dt} = rN(t, \varepsilon) \left[ 1 - \frac{N(t, \varepsilon)}{K(\varepsilon t)} \right] \\ N(0, \varepsilon) = N_0 \end{cases} \quad (6)$$

where  $r$  is a positive constant.

The solution can be represented as a function depending on “usual” time  $t_0 = t$  and “slow” time  $t_1 = \varepsilon t$ :

$$N(t, \varepsilon) \equiv n(t_0, t_1, \varepsilon) \quad (7)$$

and therefore

$$\frac{dN}{dt} = \frac{\partial n}{\partial t_0} + \varepsilon \frac{\partial n}{\partial t_1} \quad (8)$$

Substitution of the equation (8) in the equation (6) gives a multiscaled form of equation (6):

$$\begin{cases} \frac{\partial n}{\partial t_0} + \varepsilon \frac{\partial n}{\partial t_1} = rn \left[ 1 - \frac{n}{K(\varepsilon t)} \right] \\ n(0, 0, \varepsilon) = N_0 \end{cases} \quad (9)$$

Thus an ordinary differential equation is reduced to a differential equation with partial derivatives. One may solve this partial differential equation, using the method of small perturbations analysis.

Another approach for dealing with changing carrying capacity has been proposed in the earlier work of the author [Alexandrov, 2010].

The differential equation describing the logistic growth with changing carrying capacity

$$\frac{dy}{dt} = ry \left( 1 - \frac{y}{K(t)} \right) \quad (10)$$

is a special case of the Bernoulli equation which has the form

$$g(t)y_t' = f_1(t)y + f_n(t)y^n \quad (11)$$

where

$$g(t) = 1, \quad f_1(t) = r, \quad f_n(t) = -r/K(t), \quad n = 2 \quad (12)$$

The solution of this equation is known [Zaitsev and Polyanin, 2001]:

$$y^{1-n} = Ce^F + (1-n)e^F \int e^{-F} \frac{f_n(t)}{g(t)} dt, \quad F(t) = (1-n) \int \frac{f_1(t)}{g(t)} dt \quad (13)$$

Substitution of the values from equation (12) in equation (13) gives

$$y^{-1} = Ce^F - e^F \int e^{-F} \frac{K(t)}{r} dt, \quad F(t) = - \int r dt = -rt \quad (14)$$

Let us approximate  $1/K(t)$  by a trigonometric polynomial:

$$K(t) = \frac{1}{a_0 + \sum_{n=1}^N A_n(t) + B_n(t)} \quad (15)$$

$$A_n(t) = a_n \cos\left(\frac{n\pi}{L}(t - t_0)\right) \quad (16)$$

$$B_n(t) = b_n \sin\left(\frac{n\pi}{L}(t - t_0)\right) \quad (17)$$

Then, substitution of the equations (15), (16), (17) in the equation (14) gives

$$y = \frac{1}{\left(\frac{1}{y_0} - \frac{1}{K(t_0)}\right) e^{-rt} + a_0 + \sum_{n=1}^N \tilde{A}_n(t) + \tilde{B}_n(t)} \quad (18)$$

where

$$\tilde{A}_n(t) = a_n \frac{Lr \left[ Lr \cos\left(\frac{\pi n(t - t_0)}{L}\right) + \pi n \sin\left(\frac{\pi n(t - t_0)}{L}\right) \right]}{L^2 r^2 + \pi^2 n^2} \quad (19)$$

$$\tilde{B}_n(t) = b_n \frac{Lr \left[ Lr \sin\left(\frac{\pi n(t - t_0)}{L}\right) + \pi n \cos\left(\frac{\pi n(t - t_0)}{L}\right) \right]}{L^2 r^2 + \pi^2 n^2} \quad (20)$$

and  $L$  is a length of time interval at which the process is observed.

This method has its own advantages and disadvantages. If  $K(t)$  is given, then one may immediately calculate  $y(t)$ . Thus this method is very suitable for making predictions based on the scenarios of  $K(t)$  changes. At the same time, the analytical solution is the function that has much more parameters than the differential equation itself. If  $K(T)$  should be discovered from the analysis of the time-series, the problem of finding all the parameters could be computationally hard.

The following method could be more suitable for discovering the pattern of carrying capacity changes proceeding from the growth curve.

Let us approximate  $y(t)$  by a polynomial

$$y = \sum_{i=0}^L c_i t^i \quad (21)$$

where  $L$  is the length of the period of observations in years (or other appropriate units of time).

Substitution of the equation (21) in the equation (10) gives

$$\frac{d}{dt} \sum_{i=0}^L c_i t^i = r \sum_{i=0}^L c_i t^i \left( 1 - \frac{\sum_{i=0}^L c_i t^i}{K(t)} \right) \quad (22)$$

and after some algebraic manipulation one may come to the following equation for the carrying capacity

$$K(t) = \frac{r \left( \sum_{i=0}^L c_i t^i \right)^2}{r \sum_{i=0}^L c_i t^i - \sum_{i=0}^L i c_i t^{i-1}} \quad (23)$$

### 3 THE CHANGES IN THE LIMITS TO CO<sub>2</sub> EMISSIONS GROWTH

Applying the proposed method to the data on CO<sub>2</sub> emissions growth [Marland et al., 2010] one may see (Fig 1) the apparent changes in the limits to their growth.

The carrying capacity for CO<sub>2</sub> emissions resulted from liquid fossil fuels burning apparently changed from 500 MtC/y in 1930-50 to 3000 MtC/y in 1960-2000. The growth of this type of emissions seemingly represents the bi-logistic pattern of growth which is normally observed when a new technology dramatically increases availability of a limited resource.

One may also detect three “steps” in the growth of carrying capacity for CO<sub>2</sub> emissions resulted from gas burning: 750 MtC/y in 1965-75, 1250 MtC/y in 1990-2000, and 1500 MtC/y in 2003-2007.

As it can be seen from the Figure 2, the changes in carrying capacity for liquid fossil fuels correlate negatively with those for solid fossil fuels in 1980-2000. One may also see here the significant “jump” in carrying capacity for solid fossil fuels in 2000-10.

### 4 CONCLUSIONS

Addressing the problem of *managing resources of a limited planet* one should pay attention to the fact that limits to growth are changing with new technologies. The method presented in this paper is intended for discovering such changes through analysing growth curves. Application of this method to the curves displaying the growth of CO<sub>2</sub> emissions [Raupach et al, 2007] detects the pattern of changes in the limits to growth that can be interpreted as the invention of technologies that increase availability of fossil fuels. The sudden rise of CO<sub>2</sub> emissions from fossil fuels burning started in 2008 recalls the question posed by Ausubel [1976], “Can technology spare the Earth?”. The temporal pattern of carrying capacity of the carbon-based energy system staying behind the time series of global CO<sub>2</sub> emissions may reflect the temporal pattern of the trade-off between climate regulations that reduce consumption of fossil fuels and technological advances that expand their availability.

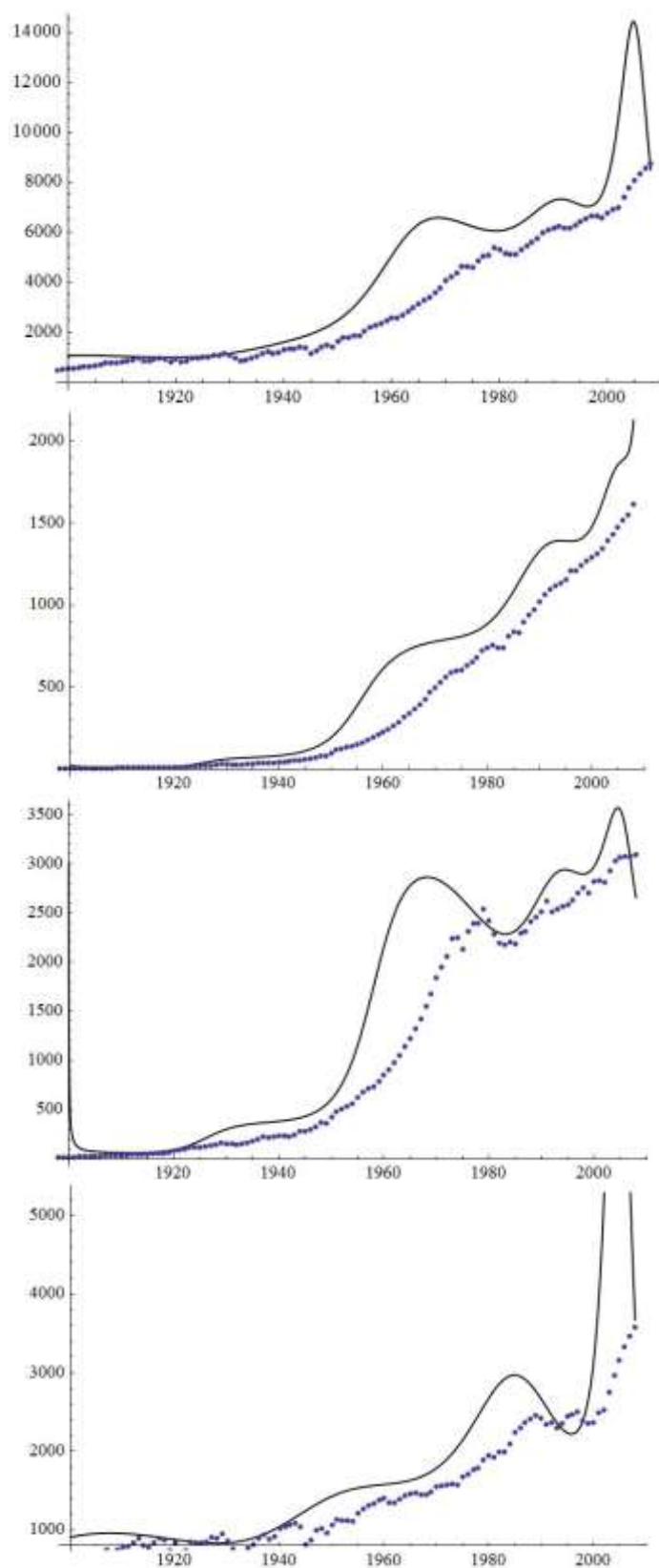


Figure 1. CO<sub>2</sub> emissions (dotted lines) from the burning all, gas, liquid, and solid fossil-fuels, respectively (from top to the bottom; the dynamics of carrying capacities are shown by solid lines). Units: MtC/y.

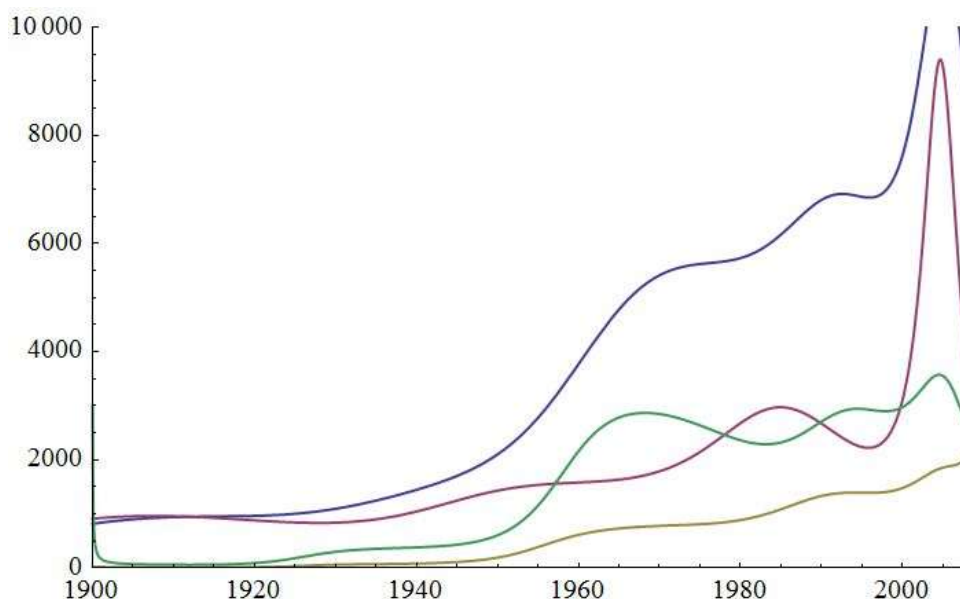


Figure 2. Carrying capacities for the emissions from burning gas (brown line), liquid fossil fuels (green line), solid fossil fuels (purple line), and all fossil fuels (blue line). Units: MtC/y.

## REFERENCES

- Alexandrov, G., Developing a model for detecting growth pulses in the observations and scenarios of CO<sub>2</sub> emissions, in: International Environmental Modelling and Software Society (iEMSs) 2010 International Congress on Environmental Modelling and Software Modelling for Environment's Sake, Fifth Biennial Meeting, Ottawa, Canada, July 5 – 8, 2010
- Ausubel, J.H., Can Technology Spare The Earth?, *American Scientist*, 84(2):166-178, 1996.
- Hui, C., Carrying capacity, population equilibrium, and environment's maximal load. *Ecological Modelling*, 192, 317-320, 2006.
- Marland, G., T.A. Boden, and R.J. Andres, Global, Regional, and National Fossil Fuel CO<sub>2</sub> Emissions, in Trends: A Compendium of Data on Global Change, Carbon Dioxide Information Analysis Center, Oak Ridge National Laboratory, U.S. Department of Energy, Oak Ridge, Tenn., U.S.A., 2010.
- Meyer, P., Bi-logistic growth. *Technological Forecasting and Social Change*, 47, 89-102, 1994.
- Raupach, M.R., G. Marland, P. Ciais, C. Le Quere, J.G. Canadell, G. Klepper, C.B. Field, Global and regional drivers of accelerating CO<sub>2</sub> emissions, *PNAS*, 104, 10288-10293, 2007.
- Shepherd, J. J. and L. Stojkov, The logistic population model with slowly varying carrying capacity, *ANZIAM J(E)* 47, C492 – C506, 2007.
- Zaitsev, V.F. and A.D. Polyanin, *Handbook of Ordinary Differential Equations*, Moscow: Fizmatlit, 2001 (in Russian).