Interacting Environmental Interfaces: Synchronization in Substance Exchange between Environmental Interfaces Regarded as Biophysical Complex Systems

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Abstract: In modeling environmental interfaces regarded as biophysical complex systems, one of the main tasks is to create an operative interface with the external environment. The interface should provide a robust and prompt translation of the vast diversity of external physical and/or chemical changes into a set of signals that are understandable for a biophysical entity. Although the organization of any system is of crucial importance for its functioning, it should not be forgotten that in biophysical systems we deal with real-life problems where a number of other conditions must be satisfied in order to put the system to work. One of them is the proper supply of the system with necessary substances. Their exchange in biophysical systems can be described by the dynamics of driven coupled oscillators. In order to study their behavior, we consider the dynamics of two coupled maps representing the substance exchange processes between two biophysical entities in their surrounding environment. Further, we investigate the behavior of the Lyapunov exponent as a measure of how rapidly two nearby orbits converge or diverge. In addition we calculate corresponding cross-sample entropy as a measure of synchronization.

Keywords: Chaos; Environmental interface; Substance exchange; Coupled logistic maps; Sample entropy.

1. INTRODUCTION

A complex system is a system composed of interacting parts that, as a whole, exhibit novel features that are usually referred to as emergent properties. A system may display one of the two forms of complexity: disorganized complexity and organized complexity [Weaver, 1948]. In disorganized complexity, the number of variables is very large and their rules of behavior are largely unknown, while organized complexity shows the essential feature of organization. Examples of complex systems include climate, populations (from simple bacterial colonies to sophisticated ant colonies), life systems and their components (e.g., the nervous system or the immune system), as well as various social structures including the economy, infrastructures, and the internet. Complex systems are studied by many areas of natural sciences, mathematics and social sciences, motivating a number of interdisciplinary investigations from diverse fields such as environmental sciences, ecology, epidemiology, cybernetics, sociology and economics [Boccara, 2004].

Until now, there has been no commonly accepted taxonomy of complex systems, but most can be classified into the following categories. (1) Chaotic systems are dynamic systems
characterized by the following properties: (i) they must be sensitive to initial conditions, (ii) they must be topologically mixing, and (iii) their periodic orbits must be dense. Sensitivity to initial conditions means that each point in such a system is arbitrarily closely approximated by other points with significantly different future trajectories. Thus, an arbitrarily small perturbation of the current trajectory may lead to a significantly different future behavior. (2) Complex adaptive systems are special cases of complex systems. They are complex in the sense that they are diverse and made up of multiple interconnected elements, and adaptive in that they have the capacity to change and learn from experience. Examples of complex adaptive systems include the stock market, social insect and ant or bacterial colonies, the biosphere and the ecosystem, the brain and the immune system, cells, development and manufacturing businesses and any human social group-based endeavors within a cultural or social system such as political parties or communities. (3) A nonlinear system is one whose behavior is not subject to the principle of superposition. Thus, if we look back at our definition we can see that a complex system is a system of interacting elements whose collective behavior cannot be described as the simple sum of the individual elements’ behaviors.

The field of environmental sciences is abundant with various interfaces and is a good place for the application of new fundamental approaches leading to a better understanding of environmental phenomena. We define the environmental interface as an interface between two abiotic or biotic environments that are in relative motion, exchanging energy through biophysical and chemical processes and fluctuating temporally and spatially regardless of the space and time scales [Mihailović and Balaž, 2007]. This definition broadly covers the unavoidable multidisciplinary approach in environmental sciences and also includes the traditional approaches in the sciences that deal with an ambient environmental space. The environmental interface as a complex system is a suitable area for the occurrence of irregularities in temporal variations of physical or biological quantities describing their interactions [van der Vaart, 1973; Varela, 1974; Rosen, 1991; Selvam, 1998; Gunji, 2006; Wolkenhauer, 2007]. For example, such an interface can be placed between cells, human or animal bodies and the surrounding environment, aquatic species and the water and air around them, and natural or artificially built surfaces and the atmosphere. Environmental interfaces, regarded as complex biophysical systems, are open and hierarchically organized, and the interactions between their parts are nonlinear, while their interaction with the surrounding environment is noisy.

In recent years, the study of deterministic mathematical models of complex biophysical systems has clearly revealed a large variety of phenomena, ranging from deterministic chaos to the presence of spatial organization. The chaos in higher dimensional systems is one of the focal subjects of physics today. Along with the approach starting from modeling physical systems with many degrees of freedom, there has emerged a new approach, developed by Kaneko [1983], in which many one-dimensional maps are coupled to study the behavior of the system as a whole. However, this model can only be applied to study the dynamics of a single medium, such as pattern formation in a fluid. What happens if two media border on each other, as at an environmental interface? One may naturally be led to a model based on coupled logistic maps with different logistic parameters. Even two logistic maps coupled to each other may serve as the dynamic model of driven coupled oscillators [Midorikawa et al., 1995]. It has been found that two identical coupled maps possess several characteristic features that are typical of higher dimensional chaos. This model of coupling can be applied, for example, to the modeling of the energy exchange between two interacting environmental interfaces [Mihailovic, 2008]. However, substance exchange in biophysical systems can be also described by the dynamics of driven coupled oscillators what will be subject of this study. In order to study their behavior, we consider the dynamics of two coupled maps representing the substance exchange processes between two biophysical entities in their surrounding environment. Further, we investigate the behavior of the Lyapunov exponent as a measure of how rapidly two nearby orbits converge or diverge. In addition we calculate corresponding cross-sample entropy as a measure of synchronization.
2. MODEL OF COUPLED MAPS REPRESENTING EXCHANGE PROCESSES BETWEEN TWO ENVIRONMENTAL INTERFACES

Following the approach introduced by Kaneko [1983], we investigate the behaviors of two interacting biophysical entities, representing environmental interfaces regarded as complex systems, when they stimulate each other in the presence of perturbations. If we denote the existence of a suitable surface as a necessary precondition that is either present or absent in the observed situation, then the other requisite points can be easily interpreted in terms of coupled logistic equations.

In order to study the model of substance exchange between two biophysical entities, we consider the dynamics of two coupled maps belonging to the same universality class as the oscillators. The system of difference equations to be investigated is of the form

\[ X_{n+1} = F(X_n) = L(X_n) + P(X_n), \]

where

\[ L(X_n) = (rx_n(1-x_n), ry_n(1-y_n)), \]

and \( X_n = (x_n, y_n) \) is a vector. For the so-called logistic parameter \( r \), which in the logistic difference equation determines an suitability of the environment and exchange processes, to we set the range from 0 to 4, and \( \varepsilon \) is a positive number in the interval (0,1]. \( P(X_n) \) represents the stimulative coupling influence of members of the system, which is here restricted only to positive numbers in the interval (0,1). The starting point \( X_0 \) is determined such that \( 0 < x_0, y_0 < 1 \). Such a choice of \( \varepsilon \) defines a model in which the mutual stimulation of growth of populations and exchange processes is obvious \( (\varepsilon < 0 \) would cause the reverse effect). In Hogg [1984], \( P(X_n) \) has the form \((\varepsilon(y_n-x_n), \varepsilon(x_n-y_n))\), with the effect that the larger population and exchange processes stimulate the growth of the smaller one, and vice versa. Some other forms of \( P(X_n) \) can be found in [Midorikawa et al., 1995]. The set \( \{(x, y): 0 < x, y < 1\} \) we denote as \( D \in \mathbb{R}^2 \), while the symmetrical of the first quadrant will be denoted as \( \Omega \). Since \( F(X) = (f(x, y), g(x, y)) \), where \( f(x, y) = rx(1-x) + \varepsilon y^p \), \( g(x, y) = ry(1-y) + \varepsilon x^p \), it is obvious that

\[ g(y, x) = f(x, y). \]
2.1 Analytical Considerations

Let us consider the starting system as \( p \to \infty \). Then, it becomes
\[
\begin{align*}
x_{n+1} &= rx_n(1-x_n), \\
y_{n+1} &= ry_n(1-y_n),
\end{align*}
\]
(4a)
(4b)
because \( x^n, y^n \to 0 \) for all \( 0 < x, y < 1 \). In that case, there is no stimulation between the two interacting biophysical entities and they behave according to the law of the logistic difference equation.

In the opposite case, as \( p \to 0 \), the starting system becomes
\[
\begin{align*}
x_{n+1} &= rx_n(1-x_n) + \varepsilon, \\
y_{n+1} &= ry_n(1-y_n) + \varepsilon,
\end{align*}
\]
(5a)
(5b)
since \( x^n, y^n \to 1 \) for all \( 0 < x, y \leq 1 \). Again, there is no interaction between the two interacting biophysical entities and again they behave according to the law of logistic equation even on the larger interval \((-\delta, 1+\delta)\), where \( \delta = \sqrt{(r-1)^2 - 4r \varepsilon - r + 1}/2r > 0 \), which is mapped onto itself under the condition \( r/4 + \varepsilon < 1+\delta \). Comparing with the standard logistic equation
\[
x_{n+1} = \rho x_n(1-x_n),
\]
(6)
we now have \( \rho = (r + 4\varepsilon + 4\delta)/(1 + 2\delta) \).

Equation \( x_{n+1} = d(x_n) \) for \( p = 1 \) becomes the logistic equation
\[
x_{n+1} = x_n(r + \varepsilon - rx_n),
\]
(7)
on the interval \((0, 1+\varepsilon/r)\), while for \( p = 2 \),
\[
x_{n+1} = x_n(r - x_n(r - \varepsilon)),
\]
(8)
which is also logistic, but now on the interval \((0, r/(r - \varepsilon))\). All of the information regarding bifurcations and chaotic behavior for Eqs. (7) and (8) is again obtained by comparing those equations with equation (6), taking \( \rho = r + \varepsilon \) and \( \rho = r \), respectively.

For the starting system, with \( p = 1 \) and \( p = 2 \) we obtain analytic expressions for fixed points and a periodic point of period two, as well as the conditions under which they are attractive. If a fixed point is in \( D \), then it must be on the diagonal. The mapping can also have fixed points off of the diagonal, but in that case, they do not belong to \( D \). Periodic points with period two that belong to \( D \) are either on the diagonal or symmetric with respect to the diagonal.

2.2 Analysis of Orbits

The orbit of the point \( X_0 \) is the sequence \( X_0, F(X_0), \ldots, F^n(X_0), \ldots \) where \( F^n(X_0) = X_0 \) and for \( n \geq 1 \), \( F^n(X_0) = F(F^{n-1}(X_0)) \). We say that the orbit is periodic with period \( k \) if \( k \) is the smallest natural number such that \( F^k(X_0) = X_0 \). If \( k = 1 \), then the point \( X_0 \) is the fixed point. The periodic point \( X_0 \) with period \( k \) is an attraction point if the norm of the Jacobi matrix for the mapping \( F^k(X) = (f_k(x, y)), (g_k(x, y)) \) is less than one, i.e., \( \|J^k(X_0)\| < 1 \), where
Here, we define $\| J^\prime(X_0) \|$ as max \{ $| \lambda_1 |, | \lambda_2 |$ \}, where $\lambda_1$ and $\lambda_2$ are the eigenvalues of the matrix. It is worth noting that

$$J^\prime(X_0) = J^\prime(X_{n-1}) \ldots J(X_{i})J(X_{i+1}),$$

where

$$J(X) = \begin{bmatrix}
    r(1-2x) & \varepsilon py^{n-1} \\
    \varepsilon px^{n-1} & r(1-2y)
\end{bmatrix}.$$ (11)

In particular, for the scalar equation $x_{n+1} = d(x_n)$ the norm is \( |(d^i(x))|_{n_{i+1}} = |d'(x)\ldots d'(x) d'(x)| \), where $d'(x) = r(1-2x) + \varepsilon px^{n-1}$. In order to characterize the asymptotic behavior of the orbits, we need to calculate the largest Lyapunov exponent, which is given for the initial point $X_0$ in the attracting region by

$$\lambda = \lim_{n \to \infty} (\ln \| J^\prime(X_0) \| / n).$$ (12)

With this exponent, we measure how rapidly two nearby orbits in an attracting region converge or diverge. In practice, using (10), we compute the approximate value of $\lambda$ by substituting in (12) successive values from $X_0$ to $X_{n_k}$, for $n_k, n_1$ large enough to eliminate transient behaviors and provide good approximation. If $X_0$ is part of a stable periodic orbit of period $k$, then $\| J^\prime(X_0) \| < 1$ and the exponent $\lambda$ is negative, which characterizes the rate

at which small perturbations from the fixed cycle decay, and we can call such a system synchronized. Quasiperiodic behavior is indicated by a zero value of $\lambda$, while Lyapunov exponent becomes positive when nearby points in the attracting region diverge from each other, indicating chaotic motion. This exponent depends on the initial point of iteration.
In order to further investigate the behavior of the coupled maps, we performed a numerical analysis of the given system. For a fixed coupling parameter $\varepsilon = 0.06$, we calculated the bifurcation diagrams for two values of $p = 0.25$ and $p = 4$ as illustrative extremes representing the influence of perturbations on the occurrence of bifurcation points (Fig. 1). As can be seen from Fig. 1, for $p = 4$, after entering the chaotic regime around $r \approx 3.5$, the appearance of a stable period four cycle can be observed. Comparing with the case $p = 0.25$, where the second appearance of a stable region is not observed, it is obvious that smaller perturbations are more favorable for the expectation of stability in the interacting system. However, this holds only within a relatively narrow range since from Eqs. (4a, and 4b) it follows that lessening the influence of perturbations will eventually lead to two independent oscillators that behave according to the law of the logistic difference equation. From Fig. 2a and 2b, which was calculated for the same parameter values, we can see that the appearance of a secondary stable period corresponding to negative values of the Lyapunov exponent. This means that within the indicated region, the introduction of small perturbation decays and the system of two interacting entities settles into a synchronized state.

![Figure 2](image_url)

**Figure 2.** Lyapunov exponent (a) and (b) and the Cross sample entropies (c) and (d) of the coupled maps as a function of logistic parameter $r$ ranging from 2.8 to 3.8. Initial conditions are $x_0 = 0.3$ and $y_0 = 0.4$.
Lyapunov exponent (Fig. 2a and 2b). Also, it should be pointed out that all calculations were done for a fixed value of $\varepsilon$. Keeping this in mind, the observed dislocation of bifurcation points with changes in the parameter $p$ indicates a region of synchronization where the emergence of order is possible.

Cross sample entropy (Cross-SampEn) measure of asynchrony is a recently introduced technique for comparing two different time series to assess their degree of asynchrony or dissimilarity. Let $u = [u(1), u(2), \ldots, u(N)]$ and $v = [v(1), v(2), \ldots, v(N)]$ fix input parameters $m$ and $r$. Vector sequences: $x(i) = [u(i), u(i+1), \ldots, u(i+m-1)]$ and $y(j) = [v(j), v(j+1), \ldots, v(j+m-1)]$ and $N$ is the number of data points of time series, $i, j = N - m + 1$. For each $i \leq N - m$ set $B^n(r)(v \parallel u) = (\text{number of } j \leq N - m \text{ such that } d[x_n(i), y_n(j)] \leq r) / (N - m)$, where $j$ ranges from 1 to $N - m$. And then

$$\begin{align*}
B^n(r)(v \parallel u) &= \sum_{i=1}^{N-m} B^n(r)(v \parallel u) / (N - m) \quad (18)
\end{align*}$$

which is the average value of $B^n(v \parallel u)$.

Similarly we define $A^n$ and $A^n$ as $A^n(r)(v \parallel u) = (\text{number of such } j \leq N - m \text{ that } d[x_n(i), y_n(j)] \leq r) / (N - m)$.

$$\begin{align*}
A^n(r)(v \parallel u) &= \sum_{i=1}^{N-m} A^n(r)(v \parallel u) / (N - m) \quad (19)
\end{align*}$$

which is the average value of $A^n(v \parallel u)$. And then

$$\begin{align*}
\text{Cross-SampEn}(m, r, n) = -\ln \left\{ A^n(r)(v \parallel u) / B^n(r)(v \parallel u) \right\} \quad (20)
\end{align*}$$

We applied Cross-SampEn with $m = 5$ and $r = 0.05$ for $x$ and $y$ time series. Figures 2c and 2d show high synchronisation between them in the interval 0.2-0.8 of coupling parameter.

4. **CONCLUSION**

In this study, we used the method of coupling one-dimensional maps to represent the dynamics of a multi-dimensional system. An equation of the form $X_{n+1} = F(X_n) = L(X_n) + P(X_n)$ where there is mutual stimulation between two interacting environmental interfaces regarded as biophysical complex systems is interpreted through substance exchange. The main finding is the observed narrowness of the synchronization region during the interactions. However, in order to specify the full region of synchronization, the influence of the attractor and its basin of attraction on the dynamics and time development in an environment of multiply changing parameters, further investigations are necessary.

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